M E T U Department of Mathematics

Math 118 Calculus II Midterm III 11.08.2017 18:00		
Last Name : Name : Student No:	Signature : Duration : 118 *minutes*	Section :
5 QUESTIONS ON 4 PAGES		TOTAL 100 POINTS
1 2 3 4 5	SHOW YOUR WORK	

1. (10+8 pts) Given $f(x,y) = \begin{cases} \dfrac{\sin(xy)}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$

(a) Find the set of points where $f(x,y)$ is continuous.

For $(x,y) \neq (0,0)$; since $x^2+y^2 \neq 0$, $f(x,y)$ is a ratio of continuous functions ($\sin(xy)$ is continuous for $\forall x,y$).

At $(x,y)=(0,0)$: $0 = f(0,0) \overset{?}{=} \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$

Along the line $y=mx$: $\lim\limits_{\substack{(x,y)\to(0,0) \\ (y=mx)}} \dfrac{\sin(xy)}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{\sin(mx^2)}{x^2+m^2x^2} = \lim\limits_{x\to 0} \dfrac{\sin(mx^2)}{x^2(m^2+1)} = \dfrac{m}{1+m^2}$

since limit depends on m, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(xy)}{x^2+y^2}$ does NOT exist

$\Rightarrow$ f is not continuous at $(0,0)$. Thus f is continuous on $\mathbb{R}^2 - \{(0,0)\}$

(b) Calculate $f_x(x,y)$ if it exists.

For $(x,y) \neq (0,0)$: $f_x(x,y) = \dfrac{y\cos(xy)(x^2+y^2) - 2x\sin(xy)}{(x^2+y^2)^2}$

At $(x,y)=(0,0)$: $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{\sin 0}{h^2} - 0}{h} = 0$

$$\Rightarrow f_x(x,y) = \begin{cases} \dfrac{y\cos(xy)(x^2+y^2) - 2x\sin(xy)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

2. (7+7 pts) Find the limit, if exists. Otherwise, show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^6+y^4}$

For all $(x,y) \neq (0,0)$, we have $y^4 \leq y^4 + x^6 \Rightarrow \dfrac{y^4}{y^4+x^6} \leq 1$.

$$0 \leq \left| x \underbrace{\frac{y^4}{x^6+y^4}}_{\leq 1} \right| \leq |x|$$

as $(x,y)\to(0,0)$: $0 \to 0$, $\left|\dfrac{xy^4}{x^6+y^4}\right| \to 0$, $|x| \to 0$ as $(x,y)\to(0,0)$

By squeeze Thm, $\displaystyle\lim_{(x,y)\to(0,0)} \left|\frac{xy^4}{x^6+y^4}\right| = 0$.

Since $-\left|\dfrac{xy^4}{x^6+y^4}\right| \leq \dfrac{xy^4}{x^6+y^4} \leq \left|\dfrac{xy^4}{x^6+y^4}\right|$

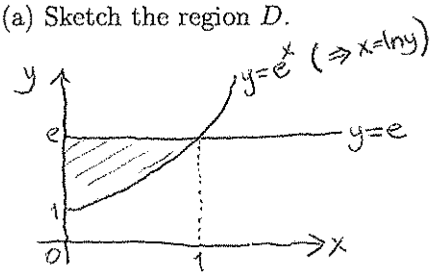
as $(x,y)\to(0,0)$, all three $\to 0$ as $(x,y)\to(0,0)$

By squeeze Thm, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^6+y^4} = 0$.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-e^{(x^2+y^2-1)}}{x^2+y^2-1} = \frac{1-e^{-1}}{-1} = \frac{1}{e} - 1$

3. (6+8 pts) Let $\quad I = \displaystyle\iint_D f(x,y)\,dA = \int_0^1 \int_{e^x}^{e} f(x,y)\,dy\,dx$ where $f(x,y) = \dfrac{y}{\ln(y)}$.

(a) Sketch the region D.



$D = \{ (x,y) : e^x \leq y \leq e, \ 0 \leq x \leq 1 \}$

(b) Rewrite I in $dx\,dy$ order and evaluate it.

$$I = \int_1^e \int_0^{\ln y} \frac{y}{\ln(y)}\,dx\,dy = \int_1^e \frac{y}{\cancel{\ln y}} \cancel{\ln y}\,dy = \left[\frac{y^2}{2}\right]_1^e = \frac{e^2-1}{2}.$$

4. (6+8+8+8 pts) Let $f(s,t)$ be a function with $f(-1,3)=0$, $f_1(-1,3)=2$ and $f_2(-1,3)=-3$.

(a) Find directional derivative of $f(s,t)$ at the point $(s,t)=(-1,3)$ in the direction of vector $\vec{u}=\langle 3,4\rangle$.

$\vec{u}=\langle 3,4\rangle$ is NOT unit $\Rightarrow$ $\frac{\vec{u}}{|\vec{u}|}=\langle \frac{3}{5},\frac{4}{5}\rangle$ is unit (as $|\vec{u}|=5$)

$$\Rightarrow D_{\vec{u}}f(-1,3)=\nabla f(-1,3)\cdot\frac{\vec{u}}{|\vec{u}|}=\langle f_1(-1,3),f_2(-1,3)\rangle\cdot\langle\frac{3}{5},\frac{4}{5}\rangle=\langle 2,-3\rangle\cdot\langle\frac{3}{5},\frac{4}{5}\rangle$$
$$=-6/5$$

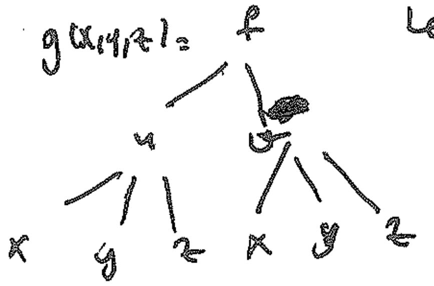
(b) Find a linear approximation for $f(s,t)$ to approximate $f(-1.05, 2.97)$.

$f(s,t)\approx L(s,t)=f(a,b)+f_1(a,b)(s-a)+f_2(a,b)(t-b)$ where $s=-1.05$, $t=2.97$, $a=-1$, $b=3$

$f(-1,3)=0$
$f_1(-1,3)=2$
$f_2(-1,3)=-3$

$$\Rightarrow f(-1.05,2.97)\cong 0+2.(-0.05)-3(-0.03)$$
$$\cong -0.1+0.09=-0.01$$

(c) Let $g(x,y,z)=f(xyz, x^2+y^2+z^2)$. Find the gradient vector $\nabla g(1,-1,1)$.



Let $u=xyz$, $v=x^2+y^2+z^2$

At $(x,y,z)=(1,-1,1)\Rightarrow (u,v)=(-1,3)$

$$\Rightarrow \nabla g(1,-1,1)=\langle g_x(1,-1,1),g_y(1,-1,1),g_z(1,-1,1)\rangle$$

$$g_x=\frac{\partial g}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\Rightarrow g_x(1,-1,1)=f_1(-1,3).(-1)+f_2(-1,3).2=-8$$

$$g_y=\frac{\partial g}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\Rightarrow g_y(1,-1,1)=f_1(-1,3).1+f_2(-1,3).(-2)=8$$

$$g_z=\frac{\partial g}{\partial z}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial z}\Rightarrow g_z(1,-1,1)=f_1(-1,3)(-1)+f_2(-1,3).2=-8$$

$$\Rightarrow \nabla g(1,-1,1)=\langle -8,8,-8\rangle$$

(d) Find an equation of tangent plane to the surface $f(xyz, x^2+y^2+z^2)=0$ at the point $(x,y,z)=(1,-1,1)$.

$\vec{n}\ //\ \nabla g=\langle -8,8,-8\rangle$

Tangent Plane equation: $-8(x-1)+8(y+1)-8(z-1)=0$

$$\Rightarrow x-y+z=3$$

5. (10+10+4 pts) Let $f(x,y) = e^{xy}$ and $D = \{(x,y) : 2x^2 + y^2 \le 1\}$.

(a) Find and classify all critical points of $f(x,y)$ in the region $\{(x,y) : 2x^2 + y^2 < 1\}$.

$f_x = ye^{xy} \Rightarrow f_x = 0 \Leftrightarrow \boxed{y=0}$

$f_y = xe^{xy} \Rightarrow f_y = 0 \Leftrightarrow \boxed{x=0}$

$\} \ \nabla f = \langle f_x, f_y \rangle = 0 \Leftrightarrow x=0, y=0$

$(0,0)$ is the only CP

$A = f_{xx} = y^2 e^{xy}$

$B = f_{xy} = e^{xy} + xye^{xy}$

$C = f_{yy} = x^2 e^{xy}$

$\}$ At $(0,0)$ $B^2 - AC = 1 > 0 \Rightarrow$ By Second Derivative Test, $(0,0)$ is a saddle point.

(b) Find the extreme values of $f(x,y)$ on the boundary $\{(x,y) : 2x^2 + y^2 = 1\}$ by using Lagrange multipliers method.

$f(x,y) = e^{xy}$, $g(x,y) = 2x^2 + y^2 = 1$, $\nabla f = \lambda \nabla g \Rightarrow \langle ye^{xy}, xe^{xy} \rangle = \lambda \langle 4x, 2y \rangle$

(1) $ye^{xy} = 4\lambda x \Rightarrow 4\lambda = \frac{y}{x} e^{xy}$

(2) $xe^{xy} = 2\lambda y \Rightarrow 2\lambda = \frac{x}{y} e^{xy}$

$\} \ \frac{2x}{y} = \frac{y}{x} \Rightarrow 2x^2 = y^2 \Leftrightarrow \boxed{2x^2 - y^2 = 0}$ (4)

(3) $2x^2 + y^2 = 1$

By (3) and (4) $\Rightarrow 4x^2 = 1 \Leftrightarrow x = \pm \frac{1}{2}$

$\boxed{\begin{array}{l} x = \frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}} \\ x = -\frac{1}{2} \Rightarrow y = \pm \frac{1}{\sqrt{2}} \end{array}}$

$\left(\frac{1}{2}, \frac{-1}{\sqrt{2}}\right), \left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

$\left(\frac{-1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{2}, \frac{-1}{\sqrt{2}}\right)$

- $f\left(\frac{1}{2}, \frac{-1}{\sqrt{2}}\right) = f\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = e^{\frac{-1}{2\sqrt{2}}}$ "the minimum of f on the boundary"
- $f\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{2}, \frac{-1}{\sqrt{2}}\right) = e^{\frac{1}{2\sqrt{2}}}$ "the maximum of f on the boundary"

(c) Explain why does $f(x,y)$ have absolute maximum and minimum on the region D? Find these values.

Since $f(x,y)$ is continuous on the closed bounded region D, by Extreme Value Theorem, it has absolute maximum and minimum on D.

$f\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{-1}{2}, \frac{-1}{\sqrt{2}}\right) = e^{\frac{1}{2\sqrt{2}}}$ absolute maximum value

$f\left(\frac{-1}{2}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{2}, \frac{-1}{\sqrt{2}}\right) = e^{\frac{-1}{2\sqrt{2}}}$ absolute minimum value

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Signature : ....